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Hierarchy of critical exponents of currents on (d + 1)-simplex fractals

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Abstract. Using the symmetry of $(d+1)$-simplex fractals with decimation number $b = 2$, the current distribution has been determined. Then using the renormalization group technique, based on the independent Schur's invariant polynomials of current distributions, the multifractal spectrum of even moments of current distributions has been evaluated analytically up to order six for an arbitrary value of d. Also the scaling exponents of order 8 and order 10 have been calculated numerically up to $d = 30$.

PACS. 64.60.Ak Renormalization-group, fractal, and percolation studies of phase transitions

1 Introduction

The study of infinite sets of exponents which originated in the field of turbulence [1] has recently become the focus of attention in a number of fields involving fractals or scaling objects [2,3] ranging from random resistor networks [4–6], dynamical systems, to diffusion limited aggregates (DLA) [7]. What is common to all of these different fields is that one wants to characterize the properties of a "weight" or a "measure" associated with different parts of a fractal object. Distribution of currents (or voltage drops) on a percolating structure in the scaling region is multifractal, in the sense that different moments scale with different exponents. That is, if we consider a system of length L, then the q-moment of the current distribution:

$$M_q = \sum_r {I_r}^q \tag{1.1}$$

scales as L^{-D_q}, where D_q is by no means a simple function of q. Thus each moment scales with its own anomalous dimension. This phenomenon is characteristic of multifractal distributions. Actually this set of exponents first appeared in the field of turbulence and has recently become the focus of attention in a number of different fields such as diffusion limited aggregation, dynamical system and random resistor networks as mentioned above. Here in this paper we study the multifractal structure of current distribution on $(d+1)$-simplex fractals, with decimation number $b = 2$. As Kirkpatrick had suggested, the so-called back-bone of the percolating random resistor networks could be modeled by a fractal structure. We note that among fractal objects, the $(d+1)$-simplex fractal is the simplest one to study the various physical problems from random walk [8–10] to electrical problem on it [5,6].

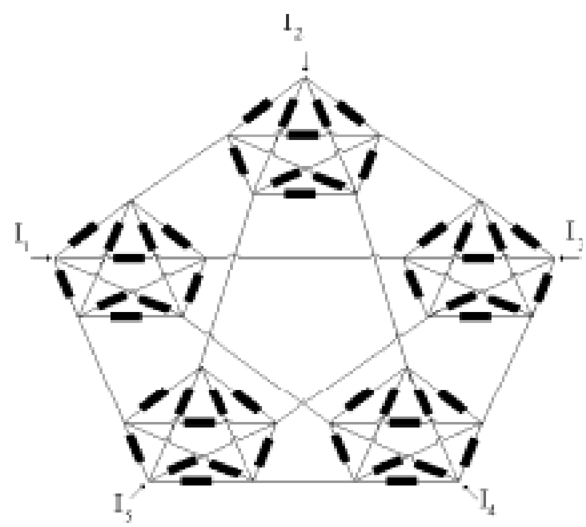

Fig. 1. Simplex fractal with decimation number $b = 2$.

Using the S_{d+1}-symmetry of $(d+1)$-simplex fractal we have been able to determine the current distribution for the decimation number $b = 2$. Then using the independent Schur's S_{d+1} invariant polynomials [11] we have evaluated D_q analytically for $q = 2$, $q = 4$ and $q = 4$ for an arbitrary value of d. The results thus obtained agree with the numerical calculation of reference [5] for $d = 2$. We have also calculated the scaling exponents of order 8 and 10 numerically up to $d = 30$.

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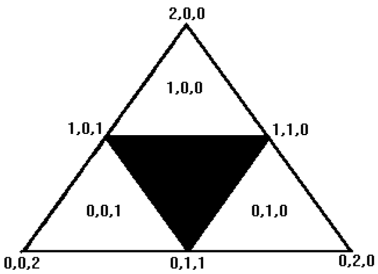

Fig. 2. Simplex fractal with decimation number $b = 2$, partitions of 1 denote the subfractals and partitions of 2 indicate the vertices, respectively.

The structure of the article is as follows: in Section 2, we give a brief description of $(d+1)$-simplex fractals with decimation number $b = 2$, in Section 3, using the S_{d+1}-symmetry of $(d+1)$-simplex fractal, we determine the inward flowing current of subfractals. In Section 4 we talk about the independent Schur's S_{d+1}-symmetry invariant polynomials of input currents. In Section 5 we give the moments of current distributions and their multifractal spectrum where the main result of this paper lies. The paper ends with a brief conclusion.

2 (d + 1)-simplex fractals with decimation number b = 2

Indeed $(d + 1)$-simplex fractal is a generalization of a 2-dimensional Sierpinski gasket to d-dimensions such that its subfractals are $(d + 1)$-simplices or d-dimensional polyhedra with S_{d+1}-symmetry. In order to obtain a fractal with decimation number $b = 2$, we choose a $(d+1)$-simplex, divide all the links (that is the lines connecting sites) into haves and then draw all possible d-dimensional hyperplanes through the links parallel to the transverse d-simplices. Next, having omitted every other inner polyhedra, we repeat this for the remaining simplices or for the subfractals of next higher order. This way through $(d+1)$-simplex fractals with fractal dimension

$$D_f = \frac{\log(d+1)}{\log 2} \tag{2.1}$$

are constructed. Although we consider a simplex in d-dimensions, we note that it is embeddable in 2 dimensions, so it can be a 2-dimensional fractal with a high fractal dimension (see Fig. 1).

3 Determination of inward flowing current of subfractals

In order to calculate the current distributions we label the subfractals of order $(l+1)$ in terms of partition of 1 into $(d+1)$ non-negative integers $\lambda_1, \lambda_2, \cdots, \lambda_{d+1}$(where one of λ's is 1 and the others are zero). Each partition represents a subfractal of order l, and λ shows the distance of the corresponding subfractal from d-dimensional hyperplanes which construct the $(d+1)$-simplex. On the other hand, each vertex, denoted by partition of 2 into $(d+1)$ non-negative integers $\eta_1, \eta_2, \cdots, \eta_{d+1}$(where at most two η are non-zero)and obviously the i-th vertex of subfractal $\lambda_1, \lambda_2, \cdots, \lambda_{d+1}$ is denoted by $\eta_j = \lambda_j + \delta_{i,j}$, where $j = 1, 2, \cdots, d+1$. As an illustration, in Figure 2 the subfractals and vertices of a 3-simplex fractal with decimation number 2 have been labelled by partition of 1 and 2 into three non-negative integers, respectively. We denote the j-th inward flowing current of subfractal corresponding to the partition $\lambda_1, \lambda_2, \cdots, \lambda_{d+1}$ by $I_{\lambda_1,\lambda_2,\cdots,\lambda_{d+1}(\lambda_1,\cdots,\lambda_{j-1},\lambda_j+1,\lambda_{j+1},\cdots,\lambda_{d+1})}$. Therefore I_j, the j-th inward flowing current of $(d+1)$-simplex fractal is given by

$$I_{0,0,\cdots,0,\underbrace{1}_{i-\mathrm{th}},0,\cdots,0}(0,0,\cdots,0,\underbrace{2}_{i-\mathrm{th}},0,\cdots,0).$$

To determine the inward flowing currents in subfractals in terms of I_j's it is enough to use the Kirchhoff's rule for each subfractal and for each vertex respectively, provided that using the S_{d+1} symmetry of $(d+1)$-symplex fractal we assume the following ansatz for the inner inward flowing currents of subfractals

$$I_{0,0,\cdots,0,\underbrace{1}_{j-\mathrm{th}},0\cdots,0}(0,0,\underbrace{1}_{j-\mathrm{th}},0,\cdots,0,\underbrace{1}_{k-\mathrm{th}},0,\cdots,0,0) = \alpha I_j + \beta I_k.$$

Then, using Kirchhoff's rule on the vertex $(0,0,\cdots,0,\underbrace{1}_{j-\mathrm{th}},0,\cdots,0,\underbrace{1}_{k-\mathrm{th}},0,\cdots,0)$, we get

$$(\alpha+\beta)(I_j + I_k) = 0.$$

That is, we have $\alpha = -\beta$. Also applying Kirchhoff's rule for the subfractal $(0,0,\cdots,0,\underbrace{1}_{j-\mathrm{th}},0,\cdots,0)$, we get

$$(1+d\alpha)I_j - \alpha\left(\sum_{k=1}^{d} I_k\right) = 0$$

hence α is equal to $-1/(d+1)$.

Finally, if we use Kirchhoff's rule for the input currents, that is $\sum_{k=1\ ,k\neq j}^{d} I_k + I_j = 0$, we get the following result for the inner currents

$$I_{0,\cdots,0,\underbrace{1}_{j-\mathrm{th}},0,\cdots,0}(0,\cdots,0,\underbrace{1}_{j-\mathrm{th}},0,\cdots,0,\underbrace{2}_{k-\mathrm{th}},0,\cdots,0) = \frac{I_k - I_j}{d+1}\cdot \tag{3.1}$$

4 Schur's polynomials of inward flowing currents

Schur's S_{d+1}-invariant polynomials [11] are homogeneous polynomials of degree q of inward flowing currents

$I_1, I_2, \cdots, I_{d+1}$:

$$S_{\lambda_1,\cdots,\lambda_{d+1}} = \sum_{\text{permutation of }(1,2,\cdots,d+1)} I_1^{\lambda_1} I_2^{\lambda_2}, \cdots, I_{d+1}^{\lambda_{d+1}} \tag{4.1}$$

where $(\lambda_1 \cdots \lambda_{d+1})$ is partition of q into $(d+1)$ non-negative integers, that is

$$\lambda_1 + \lambda_2 + \cdots + \lambda_{d+1} = q.$$

Because of the following equation due to Kirchhoff's rule

$$\sum_{k=1}^{d+1} I_k = 0$$

all of Schur's polynomials of degree q, corresponding to all possible partitions of q, are not independent. In calculating the multifractal critical exponents D_q, we must use only the independent ones. In general, the number of the invariant polynomials of degree q is:

$$P_{d+1}(q) - P_{d+1}(q-1)$$

where $P_{d+1}(q)$ is number of possible partitions of q into $(d+1)$ non-negative integers [11]. Since

$$S_1 = \sum_{k=1}^{d+1} I_k = 0 \tag{4.2}$$

then

$$S_1 S_{\lambda_1,\cdots,\lambda_{d+1}} = \sum a_{\mu_1,\cdots,\mu_{d+1}} S_{\mu_1,\cdots,\mu_{d+1}} \tag{4.3}$$

where $(\mu_1, \cdots, \mu_{d+1})$ and $(\lambda_1, \cdots, \lambda_{d+1})$ correspond to partitions of $q-1$ and q, respectively. For example, for $q = 2$, we have

$$S_1 S_1 = S_2 + 2S_{1,1} = 0 \tag{4.4}$$

therefore

$$S_{1,1} = -\frac{S_2}{2}.$$

Thus we are left with just one invariant polynomial for $q = 2$. Also for $q = 4$ we have

$$\begin{aligned} S_1 S_3 &= S_4 + S_{3,1} = 0 \\ S_1 S_{2,1} &= S_{3,1} + 2S_{2,2} + 2S_{2,1,1} = 0 \\ S_1 S_{1,1,1} &= S_{2,1,1} + 4S_{1,1,1,1} = 0 \end{aligned} \tag{4.5}$$

This means that for $d > 2$ the independent polynomials are just S_4 and $S_{2,1,1}$ and for $d = 2$, S_4 is the only independent polynomial. For $q = 6$ we have

$$\begin{aligned} S_1 S_5 &= S_6 + S_{5,1} = 0 \\ S_1 S_{4,1} &= S_{5,1} + S_{4,2} + 2S_{4,1,1} = 0 \\ S_1 S_{3,2} &= S_{4,2} + 2S_{3,3} + S_{3,2,1} = 0 \\ S_1 S_{3,1,1} &= S_{4,1,1} + S_{3,2,1} + 3S_{3,1,1,1} = 0 \\ S_1 S_{2,2,1} &= S_{3,2,1} + 2S_{2,2,1,1} + 3S_{2,2,2} = 0 \\ S_1 S_{2,1,1,1} &= S_{3,1,1,1} + 2S_{2,2,1,1} + 4S_{2,1,1,1,1} = 0 \\ S_1 S_{1,1,1,1,1} &= S_{2,1,1,1,1} + 6S_{,1,1,1,1,1,1} = 0 \end{aligned} \tag{4.6}$$

which indicate that there are only four independent invariant polynomials, namely S_6, $S_{4,2}$, $S_{3,3}$ and $S_{2,2,2}$.

Similarly, for $q = 8$ we have

$$\begin{aligned} S_1 S_7 &= S_8 + S_{7,1} = 0 \\ S_1 S_{6,1} &= S_{7,1} + S_{6,2} + 2S_{6,1,1} = 0 \\ S_1 S_{5,2} &= S_{6,2} + S_{5,3} + S_{5,2,1} = 0 \\ S_1 S_{5,1,1} &= S_{6,1,1} + S_{5,2,1} + 3S_{5,1,1,1} = 0 \\ S_1 S_{4,3} &= S_{5,3} + 2S_{4,4} + S_{4,3,1} = 0 \\ S_1 S_{4,2,1} &= S_{5,2,1} + S_{4,3,1} + 2S_{4,2,2} + 2S_{4,2,1,1} = 0 \\ S_1 S_{4,1,1,1} &= S_{5,1,1,1} + S_{4,2,1,1} + 4S_{4,1,1,1,1} = 0 \\ S_1 S_{3,3,1} &= S_{4,3,1} + S_{3,3,2} + 2S_{3,3,1,1} = 0 \\ S_1 S_{3,2,2} &= S_{4,2,2} + 2S_{3,3,2} + S_{3,2,2,1} = 0 \\ S_1 S_{3,2,1,1} &= S_{4,2,1,1} + 2S_{3,3,1,1} + 2S_{3,2,2,1} \\ &\quad + 3S_{3,2,1,1,1} = 0 \\ S_1 S_{3,1,1,1,1} &= S_{4,1,1,1,1} + S_{3,2,1,1,1} + 5S_{3,1,1,1,1,1} = 0 \\ S_1 S_{2,2,2,1} &= S_{3,2,2,1} + 4S_{2,2,2,2} + 2S_{2,2,2,1,1} = 0 \\ S_1 S_{2,2,1,1,1} &= S_{3,2,1,1,1} + S_{2,2,2,1,1} + 4S_{2,2,1,1,1,1} = 0 \\ S_1 S_{2,1,1,1,1,1} &= S_{3,1,1,1,1,1} + 2S_{2,2,1,1,1,1} \\ &\quad + 6S_{2,1,1,1,1,1,1} = 0 \\ S_1 S_{1,1,1,1,1,1,1} &= S_{2,1,1,1,1,1,1} + 8S_{1,1,1,1,1,1,1,1} = 0. \end{aligned}$$

Hence there are only seven independent invariant polynomials, namely S_8, $S_{6,2}$, $S_{5,3}$, $S_{4,4}$, $S_{4,2,2}$, $S_{3,3,2}$ and $S_{2,2,2,2}$.

Finally, for $q = 10$ we have

$$\begin{aligned} S_1 S_9 &= S_{10} + S_{9,1} = 0 \\ S_1 S_{8,1} &= S_{9,1} + S_{8,2} + 2S_{8,1,1} = 0 \\ S_1 S_{7,2} &= S_{8,2} + S_{7,3} + S_{7,2,1} = 0 \\ S_1 S_{7,1,1} &= S_{8,1,1} + S_{7,2,1} + 3S_{7,1,1,1} = 0 \\ S_1 S_{6,3} &= S_{7,3} + S_{6,4} + S_{6,3,1} = 0 \\ S_1 S_{6,2,1} &= S_{7,2,1} + S_{6,3,1} + 2S_{6,2,2} + 2S_{6,2,1,1} = 0 \\ S_1 S_{6,1,1,1} &= S_{7,1,1,1} + S_{6,2,1,1} + 4S_{6,1,1,1,1} = 0 \\ S_1 S_{5,4} &= S_{6,4} + 2S_{5,5} + S_{5,4,1} = 0 \\ S_1 S_{5,3,1} &= S_{6,3,1} + S_{5,4,1} + S_{5,3,2} + 2S_{5,3,1,1} = 0 \\ S_1 S_{5,2,2} &= S_{6,2,2} + S_{5,3,2} + S_{5,2,2,1} = 0 \\ S_1 S_{5,2,1,1} &= S_{6,2,1,1} + S_{5,3,1,1} + 2S_{5,2,2,1} \\ &\quad + 3S_{5,2,1,1,1} = 0 \end{aligned}$$

$$\begin{aligned}
S_1S_{5,1,1,1,1} &= S_{6,1,1,1,1} + S_{5,2,1,1,1} + 5S_{5,1,1,1,1,1} = 0\\
S_1S_{4,4,1} &= S_{5,4,1} + S_{4,4,2} + 2S_{4,4,1,1} = 0\\
S_1S_{4,3,2} &= S_{5,3,2} + 2S_{4,4,2} + 2S_{4,3,3} + S_{4,3,2,1} = 0\\
S_1S_{4,3,1,1} &= S_{5,3,1,1} + 2S_{4,4,1,1} + S_{4,3,2,1}\\
&\quad + 3S_{4,3,1,1,1} = 0\\
S_1S_{4,2,2,1} &= S_{5,2,2,1} + 2S_{4,3,2,1} + 3S_{4,2,2,2}\\
&\quad + 2S_{4,2,2,1,1} = 0\\
S_1S_{4,2,1,1,1} &= S_{5,2,1,1,1} + S_{4,3,1,1,1} + 2S_{4,2,2,1,1}\\
&\quad + 4S_{4,2,,1,1,1,1} = 0\\
S_1S_{4,1,1,1,1,1} &= S_{5,1,1,1,1,1} + S_{4,2,1,1,1,1}\\
&\quad + 6S_{4,1,1,1,1,1,1} = 0\\
S_1S_{3,3,3} &= S_{4,3,3} + S_{3,3,3,1} = 0\\
S_1S_{3,3,2,1} &= S_{4,3,2,1} + 3S_{3,3,3,1} + 2S_{3,3,2,2}\\
&\quad + 2S_{3,3,2,1,1} = 0\\
S_1S_{3,3,1,1,1} &= S_{4,3,1,1,1} + S_{3,3,2,1,1} + 4S_{3,3,1,1,1,1} = 0\\
S_1S_{3,2,2,2} &= S_{4,2,2,2} + 2S_{3,3,2,2} + S_{3,2,2,2,1} = 0\\
S_1S_{3,2,2,1,1} &= S_{4,2,2,1,1} + 2S_{3,3,2,1,1} + 3S_{3,2,2,2,1}\\
&\quad + 3S_{3,2,2,1,1,1} = 0\\
S_1S_{3,2,1,1,1,1} &= S_{4,2,1,1,1,1} + 2S_{3,3,1,1,1,1} + 2S_{3,2,2,1,1,1}\\
&\quad + 5S_{3,2,1,1,1,1,1} = 0\\
S_1S_{3,1,1,1,1,1,1} &= S_{4,1,1,1,1,1,1} + S_{3,2,1,1,1,1,1}\\
&\quad + 7S_{3,1,1,1,1,1,1,1} = 0\\
S_1S_{2,2,2,2,1} &= S_{3,2,2,2,1} + 5S_{2,2,2,2,2} + 2S_{2,2,2,2,1,1} = 0
\end{aligned}$$

$$\begin{aligned}
S_1S_{2,2,2,1,1,1} &= S_{3,2,2,1,1,1} + 4S_{2,2,2,2,1,1}\\
&\quad + 4S_{2,2,2,1,1,1,1} = 0\\
S_1S_{2,2,1,1,1,1,1} &= S_{3,2,1,1,1,1,1} + 3S_{2,2,2,1,1,1,1}\\
&\quad + 6S_{2,2,1,1,1,1,1,1} = 0\\
S_1S_{2,1,1,1,1,1,1,1} &= S_{3,1,1,1,1,1,1,1} + 2S_{2,2,1,1,1,1,1,1}\\
&\quad + 8S_{2,1,1,1,1,1,1,1,1} = 0\\
S_1S_{1,1,1,1,1,1,1,1,1} &= S_{2,1,1,1,1,1,1,1,1}\\
&\quad + 10S_{1,1,1,1,1,1,1,1,1,1} = 0.
\end{aligned}$$

Therefore there are only twelve independent invariant polynomials, namely S_{10}, $S_{8,2}$, $S_{7,3}$, $S_{6,4}$, $S_{6,2,2}$, $S_{5,5}$, $S_{5,3,2}$, $S_{4,4,2}$, $S_{4,3,3}$, $S_{4,2,2,2}$, $S_{3,3,2,2}$ and $S_{2,2,2,2,2}$.

5 Moments of current distribution and multifractal spectrum

From the S_{d+1} symmetry of $(d+1)$-simplex fractals, it is clear that the q-moments depend only on the independent Schur's S_{d+1} invariant polynomials of degree q of the input currents defined in Section 4, that is

$$M_q(l+1) = \sum_{\substack{\text{partitions}\\ \text{corresponding to}\\ \text{independent polynomials}}} A_{\lambda_1,\ldots,\lambda_{d+1}}(l+1)S_{\lambda_1,\cdots,\lambda_{d+1}}(l+1), \tag{5.1}$$

where $A_{\lambda_1,\cdots,\lambda_{d+1}}$ are some constants. On the other hand, $M_q(l+1)$ can be written in terms of the invariant polynomials of its l-th level subfractals, that is

$$M_q(l) = \sum_{\substack{\text{partitions}\\ \text{corresponding to}\\ \text{invariant polynomials}}} A_{\lambda_1,\ldots,\lambda_{d+1}}(l)S_{\lambda_1,\cdots,\lambda_{d+1}}(l). \tag{5.2}$$

Comparing expressions (5.1) and (5.2) we obtain the recursion relations between $A_{\lambda_1,\cdots,\lambda_{d+1}}(l)$ and $A_{\lambda_1,\cdots,\lambda_{d+1}}(l+1)$. Now, the scaling factor is defined as:

$$\lambda_q = \lim_{l\longrightarrow\infty} \frac{M_q(l+1)}{M_q(l)}\,. \tag{5.3}$$

Obviously λ_q is the maximum eigenvalue of the matrix that connects $A(l)$ and $A(l+1)$. The multifractal scaling exponent D_q is defined as:

$$D_q = \frac{\log\lambda_q}{\log b} \tag{5.4}$$

since $M_q(l)$ scale as:

$$\lim_{l\longrightarrow\infty} M_q(l) = L_l^{D_q} \tag{5.5}$$

where $L_l = b^l$. First we obtain D_2, the power scaling exponent of $(d+1)$-simplex fractals. As explained in Section 4, there are only two Schur's invariant polynomials if $q=2$. These are, S_2 and $S_{1,1}$, of which according to relation (4.7) only one, say S_2, is independent. Therefore the total power is proportional to S_2, that is

$$P(l+1) = A_2(l+1)S_2(l+1).$$

With the prescription described above, one can now show that:

$$\frac{A_2(l+1)}{A_2(l)} = 1 + \frac{2}{d+1}\cdot$$

Hence the power scaling exponent is

$$D_2 = \frac{\log\frac{d+3}{d+1}}{\log 2}\,. \tag{5.6}$$

For D_4, we have to consider S_4, $S_{3,1}$, $S_{2,2}$, $S_{2,1,1}$ and $S_{1,1,1,1}$ but due to the relations (4-9) only two of them, say S_4, $S_{2,2}$ are independent. Again, we can write:

$$M_4(l+1) = A_4(l+1)S_4(l) + A_{2,2}(l+1)S_{2,2}(l) \tag{5.7}$$

where after some calculations we obtain the following recursion relation

$$\begin{pmatrix} A_4(l+1)\\ A_{2,2}(l+1)\end{pmatrix} = \begin{pmatrix} \frac{d^4+4d^3+6d^2+6d+9}{(d+1)^4} & \frac{12}{(d+1)^4}\\ \frac{2d^3+9d^2+9d+8}{2(d+1)^4} & \frac{2d^2+7d-1}{(d+1)^4}\end{pmatrix} \times \begin{pmatrix} A_4(l)\\ A_{2,2}(l)\end{pmatrix}. \tag{5.8}$$

The largest eigenvalue of the 2×2 matrix, given above, is:

$$\lambda_{max} = \frac{\sqrt{d^8 + 8d^7 + 24d^6 + 30d^5 + 28d^4 + 120d^3 + 297d^2 + 196d + 292} + (d+1)(d^3 + 3d^2 + 5d + 8)}{2(d+1)^4} \,.$$

Hence D_4, the fourth scaling exponent is:

$$D_4 = \frac{\log\left(\frac{\sqrt{d^8 + 8d^7 + 24d^6 + 30d^5 + 28d^4 + 120d^3 + 297d^2 + 196d + 292} + (d+1)(d^3 + 3d^2 + 5d + 8)}{2(d+1)^4}\right)}{\log 2} \,. \tag{5.9}$$

For the D_6 case, we have to consider S_6, $S_{5,1}$, $S_{4,2}$, $S_{4,1,1}$, $S_{3,3}$, $S_{3,2,1}$, $S_{3,1,1,1}$, $S_{2,2,2}$, $S_{2,2,1,1}$, $S_{2,1,1,1,1}$ and $S_{1,1,1,1,1,1}$ which again due to relations (4.10) only four of them, say S_6, $S_{4,2}$, $S_{3,3}$ and $S_{2,2,2}$ are independent. After some calculations, we get

$$\begin{pmatrix} A_6(l+1) \\ A_{4,2}(l+1) \\ A_{3,3}(l+1) \\ A_{2,2,2}(l+1) \end{pmatrix} = \begin{pmatrix} \frac{d^6+6d^5+15d^4+20d^3+15d^2+8d+13}{(d+1)^6} & \frac{30}{(d+1)^6} & \frac{-40}{(d+1)^6} & 0 \\ \frac{d^5+6d^4+15d^3+23d^2+23d+6}{(d+1)^6} & \frac{d^4+4d^3+13d^2+27d+5}{(d+1)^6} & \frac{-8(d^2+3d-3)}{(d+1)^6} & \frac{36}{(d+1)^6} \\ \frac{-(2d^4+12d^3+23d^2+15d+3)}{2(d+1)^6} & \frac{-3d(2d^2+6d+5)}{2(d+1)^6} & \frac{2d^3+6d^2+5d+21}{(d+1)^6} & \frac{27}{(d+1)^6} \\ \frac{3d^4-2d^3+6d^2+23d+14}{6(d+1)^6} & \frac{2d^3+3d^2-5d-4}{2(d+1)^6} & \frac{4(3d^2+9d-4)}{3(d+1)^6} & \frac{3d^2+10d-11}{(d+1)^6} \end{pmatrix} \begin{pmatrix} A_6(l) \\ A_{4,2}(l) \\ A_{3,3}(l) \\ A_{2,2,2}(l) \end{pmatrix} \tag{5.10}$$

Again the largest eigenvalue of the 4×4 matrix is

$$\lambda_{max} = \frac{\sqrt{2}\sqrt{C-4(d+1)\sqrt{X^2(d+1)^{22}-4B}-A(d+1)\sqrt{4X(d+1)^{12}+E}-2X(d+1)^{12}}+\sqrt{4X(d+1)^{12}+E}-A(d+1)}{4(d+1)^6} \,.$$

Thus D_6, the sixth scaling exponent is

$$D_6 = \frac{\log\left(\frac{\sqrt{2}\sqrt{C-4(d+1)\sqrt{X^2(d+1)^{22}-4B}-A(d+1)\sqrt{4X(d+1)^{12}+E}-2X(d+1)^{12}}+\sqrt{4X(d+1)^{12}+E}-A(d+1)}{4(d+1)^6}\right)}{\log 2} \tag{5.11}$$

where A, B, C, E and X are functions of d:

$$A = -d^5 - 5d^4 - 11d^3 - 15d^2 - 2(11d + 14)$$

$$B = 6d^{13}+86d^{12}+495d^{11}+1303d^{10}+344d^9-8527d^8-29418d^7-52454d^6-55430d^5-29148d^4-6245d^3 - 12733d^2 - 40832d - 2967$$

$$C = d^{12}+12d^{11}+66d^{10}+220d^9+496d^8+804d^7+1018d^6+1286d^5+2123d^4+3566d^3+4176d^2+2556d+804$$

$$E = d^{12}+12d^{11}+64d^{10}+196d^9+350d^8+232d^7-480d^6-1288d^5-619d^4+1976d^3+3780d^2+2312d+824$$

$$X = \frac{2\sqrt{Q}}{3(d+1)^{12}} \cos\left(\frac{\arccos\left(\frac{R}{2\sqrt{Q^3}}\right)}{3} + \frac{4\pi}{3}\right) + \frac{F}{3(d+1)^{12}} \,. \tag{5.12}$$

Also Q, R and F are functions of d defined as

$$Q = d^{20} + 18d^{19} + 167d^{18} + 1133d^{17} + 6353d^{16} + 29498d^{15} + 109616d^{14} + 316678d^{13} + 689668d^{12} + 1058140d^{11} + 881806d^{10} - 488891d^9 - 2931493d^8 - 4842492d^7 - 4053271d^6 - 443760d^5 + 2999385d^4 + 3282948d^3 + 1542736d^2 + 473648d + 156352.$$

$$R = 2d^{30} + 54d^{29} + 717d^{28} + 6531d^{27} + 48447d^{26} + 317892d^{25} + 1843745d^{24} + 9100299d^{23} + 36991860d^{22} + 121254158d^{21} + 313851324d^{20} + 615006183d^{19} + 799891480d^{18} + 226472538d^{17} - 2075694873d^{16} - 6781276491d^{15} - 13505292270d^{14} - 20993522700d^{13} - 28589701437d^{12} - 36960656442d^{11} - 45679556553d^{10} - 50000690826d^{9} - 43576727820d^{8} - 27293058456d^{7} - 11623319646d^{6} - 5139481140d^{5} - 4796589744d^{4} - 3811789424d^{3} - 1550363856d^{2} - 162694464d + 70150336.$$

$$F = d^{10} + 12d^9 + 73d^8 + 286d^7 + 749d^6 + 1287d^5 + 1371d^4 + 795d^3 + 198d^2 + 122d - 10.$$

For D_8, due to the relations (4-7) among the Schur's invariant polynomials of degree eight, only seven of them, say S_8, $S_{6,2}$, $S_{5,3}$, $S_{4,4}$, $S_{4,2,2}$, $S_{3,3,2}$ and $S_{2,2,2,2}$ are independent. After some calculations, we get the following expression for the 7×7 connecting matrix

$$\begin{pmatrix} \begin{pmatrix} A_8(l+1) \\ A_{6,2}(l+1) \\ A_{5,3}(l+1) \\ A_{4,4}(l+1) \end{pmatrix} \\ \begin{pmatrix} A_{4,2,2}(l+1) \\ A_{3,3,2}(l+1) \\ A_{2,2,2,2}(l+1) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} B_{1,1}(4\times 3) & B_{1,2}(4\times 3) \\ B_{2,1}(3\times 4) & B_{2,2}(3\times 3) \end{pmatrix} \begin{pmatrix} \begin{pmatrix} A_8(l) \\ A_{6,2}(l) \\ A_{5,3}(l) \\ A_{4,4}(l) \end{pmatrix} \\ \begin{pmatrix} A_{4,2,2}(l) \\ A_{3,3,2}(l) \\ A_{2,2,2,2}(l) \end{pmatrix} \end{pmatrix}$$

where the matrices $B_{1,1}(4 \times 4)$, $B_{1,2}(4 \times 3)$, $B_{2,1}(3 \times 4)$ and $B_{2,2}(3 \times 3)$ are defined as

$$\mathrm{B}_{1,1}(4\times 4) =$$

$$\begin{pmatrix} \frac{2(d+8)}{(d+1)^8}+1 & \frac{56}{(d+1)^8} & \frac{-112}{(d+1)^8} & \frac{140}{(d+1)^8} \\ \frac{d^7+8d^6+27d^5+50d^4+56d^3+45d^2+33d+10}{(d+1)^8} & \frac{d^6+6d^5+15d^4+20d^3+31d^2+56d+19}{(d+1)^8} & \frac{-2(13d^2+39d-10)}{(d+1)^8} & \frac{10(3d^2+9d-2)}{(d+1)^8} \\ \frac{-(d^6+8d^5+26d^4+48d^3+52d^2+25d+1)}{(d+1)^8} & \frac{-(3d^5+15d^4+40d^3+60d^2+32d-4)}{(d+1)^8} & \frac{d^5+5d^4+21d^3+43d^2+27d+53}{(d+1)^8} & \frac{-10(d^3+3d^2+2d+5)}{(d+1)^8} \\ \frac{2d^5+16d^4+44d^3+57d^2+41d+16}{(2(d+1)^8)} & \frac{2(3d^4+12d^3+18d^2+15d+8)}{(d+1)^8} & \frac{-4(d^4+4d^3+6d^2+5d-4)}{(d+1)^8} & \frac{2d^4+8d^3+12d^2+11d-17}{(d+1)^8} \end{pmatrix},$$

$$\mathrm{B}_{1,2}(4\times 3) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{60}{(d+1)^8} & \frac{-40}{(d+1)^8} & 0 \\ \frac{90}{(d+1)^8} & \frac{-60}{(d+1)^8} & 0 \\ \frac{48}{(d+1)^8} & -32/(d+1)^8 & 0 \end{pmatrix},$$

$$\mathrm{B}_{2,1}(3\times 4) = \begin{pmatrix} \frac{d^6+7d^5+20d^4+37d^3+44d^2+19d-4}{(2(d+1)^8)} & \frac{d^5+3d^4+9d^3+17d^2+2d-12}{(d+1)^8} & \frac{2(d^4+2d^3+8d^2+25d-2)}{(d+1)^8} & \frac{2d^3-9d^2-41d+20}{(d+1)^8} \\ \frac{-(2d^5+13d^4+24d^3+13d^2+4d+7)}{(2(d+1)^8)} & \frac{-2(2d^4+4d^3-d^2-d+5)}{(d+1)^8} & \frac{d^4-6d^3-17d^2+6d-16}{(d+1)^8} & \frac{4d^3+6d^2-10d+35}{(d+1)^8} \\ \frac{d(4d^4+21d^3+14d^2-49d-54)}{(24(d+1)^8)} & \frac{3d^4-2d^3-27d^2-22d+8}{(6(d+1)^8)} & \frac{6d^3+7d^2-21d-2}{(3(d+1)^8)} & \frac{4(d^2+3d-1)}{(d+1)^8} \end{pmatrix},$$

$$\mathrm{B}_{2,2}(3\times 3) = \begin{pmatrix} \frac{d^4+4d^3+20d^2+48d-41}{(d+1)^8} & \frac{-8(d^2+3d-7)}{(d+1)^8} & \frac{72}{(d+1)^8} \\ \frac{-3(2d^3+3d^2-5d+16)}{(d+1)^8} & \frac{2d^3+d^2-10d+71}{(d+1)^8} & \frac{108}{(d+1)^8} \\ \frac{2d^3-3d^2-23d+10}{(2(d+1)^8)} & \frac{4(3d^2+9d-10)}{(3(d+1)^8)} & \frac{4d^2+13d-27}{(d+1)^8} \end{pmatrix}. \quad (5.13)$$

Unlike the three previous cases, it is not posssible to calculate analytically the largest eigenvalue of this 7×7 transfer matrix. Hence we have diagonalised it numerically up to d=30 and the results obtained for D_8 are given in Table 1.

Finally, for the D_{10}, because of the relations (4-8) among the Schur's invariant polynomials of degree ten, only twelve of them say S_{10}, $S_{8,2}$, $S_{7,3}$, $S_{6,4}$, $S_{6,2,2}$, $S_{5,5}$, $S_{5,3,2}$, $S_{4,4,2}$, $S_{4,3,3}$, $S_{4,2,2,2}$, $S_{3,3,2,2}$ and $S_{2,2,2,2,2}$ are independent. After some tedious calculations, we get the following expression for the 12×12 connecting matrix

$$\begin{pmatrix} \begin{pmatrix} A_{10}(l+1) \\ A_{8,2}(l+1) \\ A_{7,3}(l+1) \end{pmatrix} \\ \begin{pmatrix} A_{6,4}(l+1) \\ A_{6,2,2}(l+1) \\ A_{5,5}(l+1) \end{pmatrix} \\ \begin{pmatrix} A_{5,3,2}(l+1) \\ A_{4,4,2}(l+1) \\ A_{4,3,3}(l+1) \end{pmatrix} \\ \begin{pmatrix} A_{4,2,2,2}(l+1) \\ A_{3,3,2,2}(l+1) \\ A_{2,2,2,2,2}(l+1) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} C_{1,1}(3\times 3) & C_{1,2}(3\times 3) & C_{1,3}(3\times 3) & C_{1,4}(3\times 3) \\ C_{2,1}(3\times 3) & C_{2,2}(3\times 3) & C_{2,3}(3\times 3) & C_{2,4}(3\times 3) \\ C_{3,1}(3\times 3) & C_{3,2}(3\times 3) & C_{3,3}(3\times 3) & C_{3,4}(3\times 3) \\ C_{4,1}(3\times 3) & C_{4,2}(3\times 3) & C_{4,3}(3\times 3) & C_{4,4}(3\times 3) \end{pmatrix} \begin{pmatrix} \begin{pmatrix} A_{10}(l) \\ A_{8,2}(l) \\ A_{7,3}(l) \end{pmatrix} \\ \begin{pmatrix} A_{6,4}(l) \\ A_{6,2,2}(l) \\ A_{5,5}(l) \end{pmatrix} \\ \begin{pmatrix} A_{5,3,2}(l) \\ A_{4,4,2}(l) \\ A_{4,3,3}(l) \end{pmatrix} \\ \begin{pmatrix} A_{4,2,2,2}(l) \\ A_{3,3,2,2}(l) \\ A_{2,2,2,2,2}(l) \end{pmatrix} \end{pmatrix}$$

where the matrices $C_{1,1}(3\times 3)$, $C_{1,2}(3\times 3)$, $C_{1,3}(3\times 3)$, $C_{1,4}(3\times 3)$, $C_{2,1}(3\times 3)$, $C_{2,2}(3\times 3)$, $C_{2,3}(3\times 3)$, $C_{2,4}(3\times 3)$, $C_{3,1}(3\times 3)$, $C_{3,2}(3\times 3)$, $C_{3,3}(3\times 3)$, $C_{3,4}(3\times 3)$ $C_{4,1}(3\times 3)$, $C_{4,2}(3\times 3)$, $C_{4,3}(3\times 3)$ and $C_{4,4}(3\times 3)$ are defined as

$$C_{1,1}(3\times 3) =$$

$$\begin{pmatrix} \frac{2(d+10)}{(d+1)^{(10)}}+1 & \frac{90}{(d+1)^{(10)}} & \frac{-240}{(d+1)^{(10)}} \\ \frac{d(d^8+10d^7+44d^6+112d^5+182d^4+196d^3+141d^2+75d+43)}{(d+1)^{(10)}} & \frac{(d^8+8d^7+28d^6+56d^5+70d^4+56d^3+57d^2+111d+41)}{(d+1)^{(10)}} & \frac{-64d(d+3)}{(d+1)^{(10)}} \\ \frac{-(d^8+10d^7+42d^6+98d^5+141d^4+136d^3+93d^2+35d-1)}{(d+1)^{(10)}} & \frac{-3(d^7+7d^6+21d^5+35d^4+42d^3+42d^2+20d-5)}{(d+1)^{(10)}} & \frac{(d^7+7d^6+21d^5+35d^4+71d^3+129d^2+79d+73)}{(d+1)^{(10)}} \end{pmatrix},$$

$$C_{1,2}(3\times 3) = \begin{pmatrix} \frac{420}{(d+1)^{(10)}} & 0 & \frac{-504}{(d+1)^{(10)}} \\ \frac{14(7d^2+21d-2)}{(d+1)^{(10)}} & \frac{112}{(d+1)^{(10)}} & \frac{-56(2d^2+6d-1)}{(d+1)^{(10)}} \\ \frac{-42(d^3+3d^2+2d+3)}{(d+1)^{(10)}} & \frac{42(d^3+3d^2+3d+5)}{(d+1)^{(10)}} & \frac{-42(d-3)}{(d+1)^{(10)}} \end{pmatrix},$$

$$C_{1,3}(3\times 3) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{-112}{(d+1)^{(10)}} & \frac{140}{(d+1)^{(10)}} & 0 \\ \frac{-168}{(d+1)^{(10)}} & \frac{210}{(d+1)^{(10)}} & 0 \end{pmatrix},$$

$$C_{1,4}(3\times 3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_{2,1}(3\times 3) =$$

$$\begin{pmatrix} \frac{(d^7+10d^6+40d^5+90d^4+125d^3+107d^2+59d+24)}{(d+1)^{(10)}} & \frac{3(2d^6+12d^5+35d^4+60d^3+60d^2+39d+24)}{(d+1)^{(10)}} & \frac{-4d(d^5+6d^4+20d^3+40d^2+45d+32)}{(d+1)^{(10)}} \\ \frac{(d^8+9d^7+33d^6+65d^5+77d^4+70d^3+64d^2+33d-4)}{(2(d+1)^{(10)})} & \frac{(2d^7+10d^6+18d^5+10d^4+22d^3+79d^2+51d-36)}{(2(d+1)^{(10)})} & \frac{2(d^6+6d^5+15d^4+10d^3-d^2+29d+4)}{(d+1)^{(10)}} \\ \frac{-(2d^6+20d^5+70d^4+120d^3+109d^2+43d-5)}{(d+1)^{(10)}} & \frac{-5(4d^5+20d^4+40d^3+40d^2+16d-7)}{(d+1)^{(10)}} & \frac{20(d^5+5d^4+10d^3+10d^2+4d+2)}{(d+1)^{(10)}} \end{pmatrix},$$

$$C_{2,2}(3\times 3) =$$

$$\begin{pmatrix} \frac{(d^6+6d^5+31d^4+84d^3+111d^2+88d-49)}{(d+1)^{(10)}} & \frac{192}{(d+1)^{(10)}} & \frac{-12(d^4+4d^3+6d^2+5d-8)}{(d+1)^{(10)}} \\ \frac{2(8d^3+2d^2-50d+9}{(d+1)^{(10)}} & \frac{(d^6+6d^5+15d^4+20d^3+47d^2+104d-71)}{(d+1)^{(10)}} & \frac{-2(6d^3-9d^2-69d+10)}{(d+1)^{(10)}} \\ \frac{-10(d^5+5d^4+10d^3+10d^2+4d+7)}{(d+1)^{(10)}} & \frac{200}{(d+1)^{(10)}} & \frac{2(2d^5+10d^4+20d^3+20d^2+9d+53)}{(d+1)^{(10)}} \end{pmatrix},$$

$$C_{2,3}(3\times 3)=\begin{pmatrix} \frac{-192}{(d+1)^{(}10)} & \frac{240}{(d+1)^{(}10)} & 0 \\ \frac{-2(13d^2+39d-58)}{(d+1)^{(}10)} & \frac{10(3d^2+9d-11)}{(d+1)^{(}10)} & \frac{-20}{(d+1)^{(}10)} \\ \frac{-200}{(d+1)^{(}10)} & \frac{250}{(d+1)^{(}10)} & 0 \end{pmatrix},$$

$$C_{2,4}(3\times 3)=\begin{pmatrix} 0 & 0 & 0 \\ \frac{90}{(d+1)^{(}10)} & \frac{-40}{(d+1)^{(}10)} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_{3,1}(3\times 3)=$$

$$\begin{pmatrix} \frac{-(d^7+9d^6+32d^5+63d^4+69d^3+30d^2+9)}{(d+1)^{(}10)} & \frac{-(4d^6+19d^5+46d^4+54d^3-2d^2-31d+39)}{(d+1)^{(}10)} & \frac{(d^6-3d^5-20d^4-66d^3-102d^2+12d+4)}{(d+1)^{(}10)} \\ \frac{(4d^6+36d^5+113d^4+178d^3+162d^2+79d+4)}{(2(d+1)^{(}10))} & \frac{(28d^5+116d^4+196d^3+193d^2+105d-30)}{(2(d+1)^{(}10))} & \frac{-4(2d^5+2d^4-11d^3-28d^2-34d-11)}{(d+1)^{(}10)} \\ \frac{(d^6+7d^5+11d^4-5d^3-22d^2-20d-17)}{(2(d+1)^{(}10))} & \frac{3(2d^5+3d^4-8d^3-18d^2-15d-21)}{(2(d+1)^{(}10))} & \frac{-(d^5-8d^4-18d^3+7d^2-4d-2)}{(d+1)^{(}10)} \end{pmatrix},$$

$$C_{3,2}(3\times 3)=\begin{pmatrix} \frac{(2d^5+5d^4+27d^3+43d^2-47d+84)}{(d+1)^{(}10)} & \frac{-2(3d^5+15d^4+40d^3+30d^2-58d+75)}{(d+1)^{(}10)} & \frac{2(d^4-7d^3-14d^2+22d-56)}{(d+1)^{(}10)} \\ \frac{(2d^5-6d^4-43d^3-86d^2-100d+17)}{(d+1)^{(}10)} & \frac{4(6d^4+24d^3+45d^2+54d-7)}{(d+1)^{(}10)} & \frac{4(2d^4+8d^3+14d^2+15d-11)}{(d+1)^{(}10)} \\ \frac{-(8d^4+2d^3-43d^2-23d-112)}{(2(d+1)^{(}10))} & \frac{9(d^4-6d^2-3d-11)}{(d+1)^{(}10)} & \frac{-6(d^3+3d^2+2d+10)}{(d+1)^{(}10)} \end{pmatrix},$$

$$C_{3,3}(3\times 3)=\begin{pmatrix} \frac{(d^5+5d^4+21d^3+d^2-99d+277)}{(d+1)^{(}10)} & \frac{-10(d^3-7d+27)}{(d+1)^{(}10)} & \frac{10(d^2+3d-4)}{(d+1)^{(}10)} \\ \frac{-4(2d^4+8d^3+18d^2+27d-31)}{(d+1)^{(}10)} & \frac{(4d^4+16d^3+37d^2+58d-133)}{(d+1)^{(}10)} & \frac{8(d^2+3d-2)}{(d+1)^{(}10)} \\ \frac{-3(d^4-2d^3-12d^2-7d-58)}{(d+1)^{(}10)} & \frac{-3(2d^3+6d^2+4d+47)}{(d+1)^{(}10)} & \frac{(d^4-2d^3-12d^2-6d-39)}{(d+1)^{(}10)} \end{pmatrix},$$

$$C_{3,4}(3\times 3)=\begin{pmatrix} \frac{270}{(d+1)^{(}10)} & \frac{-120}{(d+1)^{(}10)} & 0 \\ \frac{144}{(d+1)^{(}10)} & \frac{-64}{(d+1)^{(}10)} & 0 \\ \frac{189}{(d+1)^{(}10)} & \frac{-84}{(d+1)^{(}10)} & 0 \end{pmatrix},$$

$$C_{4,1}(3\times 3)=$$

$$\begin{pmatrix} \frac{(d^7+7d^6+17d^5+24d^4+4d^3-71d^2-94d-8)}{(6(d+1)^{(}10)} & \frac{(d^6+d^5-11d^3-65d^2-86d+8)}{(2(d+1)^{(}10)} & \frac{2(3d^5+5d^4+19d^3+36d^2-43d-32)}{(3(d+1)^{(}10))} \\ \frac{-(2d^6+12d^5+9d^4-28d^3-24d^2+13d+22)}{(6(d+1)^{(}10))} & \frac{-(10d^5-2d^4-76d^3-41d^2+105d-362)}{(6(d+1)^{(}10))} & \frac{(d^5-20d^4-15d^3+68d^2-58d+160)}{(3(d+1)^{(}10))} \\ \frac{(5d^6+21d^5-40d^4-175d^3-25d^2+214d+32)}{(120(d+1)^{(}10))} & \frac{d(4d^4-15d^3-58d^2+35d+162)}{(24(d+1)^{(}10))} & \frac{(3d^4-4d^3-24d^2+d+10)}{(3(d+1)^{(}10))} \end{pmatrix},$$

$$C_{4,2}(3\times 3)=\begin{pmatrix} \frac{(15d^4+10d^3+87d^2+416d-60)}{(6(d+1)^{(}10))} & \frac{(d^5+3d^3-25d^2-127d+12)}{(2(d+1)^{(}10))} & \frac{(2d^3-3d^2-23d+14)}{(d+1)^{(}10)} \\ \frac{2(2d^4-11d^3-23d^2+29d-11)}{(3(d+1)^{(}10))} & \frac{-(14d^4-34d^3-136d^2+292d-217)}{(6(d+1)^{(}10))} & \frac{4(d^3-7d+10)}{(3(d+1)^{(}10))} \\ \frac{(6d^3+3d^2-33d+2)}{(3(d+1)^{(}10))} & \frac{(3d^4-20d^3-24d^2+113d+2)}{(12(d+1)^{(}10))} & \frac{4(5d^2+15d-9)}{(15(d+1)^{(}10))} \end{pmatrix},$$

Table 1. The largest eigenvalue of the 7×7 matrix (5-12) and the obtained eighth-order scaling exponent D_8 are given for dimensions of up to $d = 30$.

d (dimension)	λ_{max} the largest eigenvalue	D_8 the scaling exponents of order eight
2	1.01595537593912	0.02283703564829
3	1.00091708813146	0.00132247218019
4	1.00012213229371	0.00017618889551
5	1.00002560310989	0.00003693700682
6	1.00000714474686	0.00001030765404
7	1.00000242856159	0.00000350366951
8	1.00000095367897	0.00000137586726
9	1.00000041815120	0.00000060326453
10	1.00000020000020	0.00000028853926
11	1.00000010263167	0.00000014806620
12	1.00000005581634	0.00000008052596
13	1.00000003187327	0.00000004598340
14	1.00000001897290	0.00000002737211
15	1.00000001170553	0.00000001688751
16	1.00000000745058	0.00000001074892
17	1.00000000487402	0.00000000703173
18	1.00000000326680	0.00000000471299
19	1.00000000223746	0.00000000322797
20	1.00000000156250	0.00000000225421
21	1.00000000111044	0.00000000160203
22	1.00000000080181	0.00000000115677
23	1.00000000058740	0.00000000084744
24	1.00000000043607	0.00000000062911
25	1.00000000032768	0.00000000047274
26	1.00000000024901	0.00000000035925
27	1.00000000019120	0.00000000027584
28	1.00000000014823	0.00000000021385
29	1.00000000011594	0.00000000016727
30	1.00000000009145	0.00000000013193

$$C_{4,3}(3\times 3) = \begin{pmatrix} \frac{2(d^4+2d^3+20d^2+61d-42)}{(d+1)^{(}10)} & \frac{(2d^3-15d^2-59d+94)}{(2(d+1)^{(}10))} & \frac{-2(3d^2+9d-7)}{(d+1)^{(}10)} \\ \frac{(2d^4-36d^3-35d^2+177d-228)}{(3(d+1)^{(}10))} & \frac{(4d^3-6d^2-46d+157)}{(3(d+1)^{(}10))} & \frac{(10d^3+d^2-67d+124)}{(6(d+1)^{(}10))} \\ \frac{(6d^3-11d^2-75d+34)}{(3(d+1)^{(}10))} & \frac{2(d^2+3d-4)}{(d+1)^{(}10)} & \frac{2(3d^2+9d-5)}{(3(d+1)^{(}10))} \end{pmatrix},$$

and

$$C_{4,4}(3\times 3) = \begin{pmatrix} \frac{(d^4+4d^3+27d^2+69d-129)}{(6(d+1)^{(}10))} & \frac{-2(d^2+3d-11)}{(d+1)^{(}10)} & \frac{1}{(d+1)^{(}10)} & \frac{-(2d^3-14d+49)}{(2(d+1)^{(}10))} \\ & \frac{(2d^3-4d^2-25d+163)}{(6(d+1)^{(}10))} & & \frac{3}{(2(d+1)^{(}10))} \\ \frac{(2d^3-9d^2-41d+42)}{(12(d+1)^{(}10))} & \frac{(3d^2+9d-16)}{(3(d+1)^{(}10))} & & \frac{(5d^2+16d-49)}{(120(d+1)^{(}10))} \end{pmatrix}. \tag{5.13}$$

Table 2. The largest eigenvalue of the 12 × 12 matrix (5-13) and the obtained tenth-order scaling exponent D_{10} are given for dimensions of up to $d = 30$.

d (dimension)	λ_{max} the largest eigenvalue	D_{10} the scaling exponents of order ten
2	1.00450678816719	0.00648731342118
3	1.00010764130781	0.00015528522356
4	1.00000783165322	0.00001129864303
5	1.00000103734742	0.00000149657521
6	1.00000019983509	0.00000028830107
7	1.00000004975737	0.00000007178471
8	1.00000001493645	0.00000002154874
9	1.00000000517003	0.00000000745878
10	1.00000000200194	0.00000000288819
11	1.00000000084875	0.00000000122449
12	1.00000000038779	0.00000000055946
13	1.00000000018866	0.00000000027218
14	1.00000000009682	0.00000000013969
15	1.00000000005203	0.00000000007507
16	1.00000000002911	0.00000000004199
17	1.00000000001687	0.00000000002433
18	1.00000000001008	0.00000000001455
19	1.00000000000620	0.00000000000894
20	1.00000000000391	0.00000000000564
21	1.00000000000252	0.00000000000363
22	1.00000000000166	0.00000000000239
23	1.00000000000111	0.00000000000160
24	1.00000000000076	0.00000000000109
25	1.00000000000052	0.00000000000076
26	1.00000000000037	0.00000000000053
27	1.00000000000026	0.00000000000038
28	1.00000000000019	0.00000000000027
29	1.00000000000014	0.00000000000020
30	1.00000000000010	0.00000000000015

Again we have diagonalized the 12 × 12 transfer matrix numerically up to $d = 30$ and the results obtained for D_{10} is given in Table 2.

6 Conclusion

Tables 1 and 2 show that the scaling exponents of order beyond ten decreas very rapidly as d increases. Hence, one needs to calculate numerically the higher order scaling exponents only for the first few lower dimensions such as $d = 2$ and $d = 3$, which is under investigation. Also in $(d + 1)$-simplex fractals with decimation number $b \geq 3$, the symmetry of fractal cannot determine the current distributions. This means that in order to calculate the multifractal spectrum and also restoration of isotropy [12] for higher decimation numbers in an arbitrary dimension, one should combine the symmetry of fractals with the minimization of power. This is under separate analytical and numerical investigation, too.

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